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2-extendability of toroidal polyhexes and Klein-bottle polyhexes[☆]

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A B S T R A C T

A toroidal polyhex (resp. Klein-bottle polyhex) described by a string (p, q, t) arises from a $p \times q$-parallelogram of a hexagonal lattice by a usual torus (resp. Klein bottle) boundary identification with a torsion t. A connected graph G admitting a perfect matching is k-extendable if $|V(G)| \geq 2k + 2$ and any k independent edges can be extended to a perfect matching of G. In this paper, we characterize 2-extendable toroidal polyhexes and 2-extendable Klein-bottle polyhexes.

1. Introduction

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. A *matching* of G is a set of independent edges. A matching M of G is *perfect* (or *1-factor*) if M covers every vertex of G. A connected graph G is *k-extendable* if $|V(G)| \geq 2k + 2$ and any k independent edges of G belong to a perfect matching of G. A 2-extendable bipartite graph is also called a *brace*. Braces play a key role in matching theory [14]; for example, braces are fundamental blocks in the "tight edge cut decomposition" of 1-extendable graphs [13]. A generating method for all braces was given by McCuaig [16]. Robertson, Seymour and Thomas [23], and independently, McCuaig [17] presented a good characterization for the braces admitting a Pfaffian orientation (see also [18]).

The *extendability* of G is the maximum integer k such that G is k-extendable. The extendability of a bipartite graph can be determined in polynomial time [12,28]. Let G be a graph admitting a 2-cell embedding on a surface Σ. G is a *strong embedding* (or closed 2-cell embedding [19]) if every face of G is bounded by a cycle. If Σ is a surface other than the sphere, Dean [3] proved that the extendability of G is no more than $1 + \lfloor\sqrt{4 - 2\chi}\rfloor$, where χ is the Euler characteristic of Σ. According to Dean's result, the extendability of a graph embeddable on the torus or the Klein-bottle is at most 3. For a planar graph G, its extendability is at most 2 [22]. A *fullerene graph* is a cubic 3-connected plane graph with 12 faces with size 5 and other faces of size 6. By a result of Holton and Plummer concerning cubic 3-connected plane graphs [5], Zhang and Zhang [30] pointed out that the extendability of a fullerene graph is 2.

A fullerene graph corresponds to a spherical fullerene molecule in chemistry. After the discovery of spherical fullerenes, the extension of fullerenes on other surfaces was considered [4]. The torus and the Klein bottle are the only two surfaces that can be tiled entirely with hexagons; the corresponding tilings are called toroidal polyhex and Klein-bottle polyhex [4,8], respectively, also "elemental benzenoids" [10]. We may refer to [7,9] for a comprehensive discussion on fullerene structures. A perfect matching of a graph coincides with a Kekulé structure of the corresponding organic molecule. Kekulé structures play an important role in resonance theory and valence bond theory. There is a series of work enumerating perfect matchings

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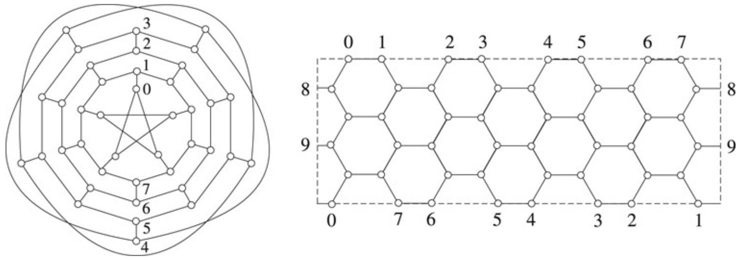

Fig. 1. A non-bipartite Klein-bottle polyhex and its rectangular representation.

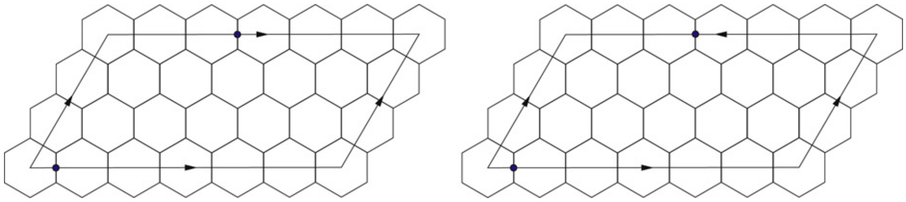

Fig. 2. A 6×3-parallelogram in a hexagonal lattice, the torus boundary identification (left) and the Klein bottle boundary identification (right) ($t = 2$).

of toroidal polyhexes and Klein-bottle polyhexes [2,6,11]. Recently, k-resonant toroidal polyhexes and k-resonant Klein-bottle polyhexes were characterized [25,26].

A toroidal polyhex can be considered as a hexagonal tessellation (or dually triangulations) of the torus [20,27] determined by a unique string of three integers [1]. Toroidal polyhexes are bipartite and cover many interesting graphs, such as $K_{3,3}$, Cube (Q_3), Heawood graph, generalized Petersen graph $G(8, 3)$ [15,24] and some circulant graphs [29]. A Klein-bottle polyhex is treated as a hexagonal tessellation of the Klein bottle. Thomassen [27] classified Klein-bottle polyhexes into five classes. Klein-bottle polyhexes considered here are bipartite and can be analogously defined as toroidal polyhexes (detailed definitions are deferred to Section 2). Other Klein-bottle polyhexes are non-bipartite. An example of non-bipartite Klein-bottle polyhexes is shown in Fig. 1.

Both a toroidal polyhex and a Klein-bottle polyhex are 1-extendable [25,26]. But they are not 3-extendable since a k-extendable graph is $(k+1)$-connected [21]. In this paper, we consider the 2-extendability of toroidal polyhexes and Klein-bottle polyhexes and obtain the following main theorem.

Theorem 1.1. *A toroidal polyhex (resp. Klein-bottle polyhex) is a brace if and only if it is a strong embedding.* □

Theorem 1.1 implies that all strong embeddable toroidal polyhexes and Klein-bottle polyhexes are 2-extendable.

2. Preliminaries

Let P be a $p \times q$-parallelogram in a hexagonal lattice as illustrated in Fig. 2: every corner of P lies at the center of a hexagon, the top side is parallel to the bottom side across p vertical edges and the two parallel lateral sides pass through q edges perpendicular to them. A *toroidal polyhex* $H(p, q, t)$ is obtained from P with a torus boundary identification: first identify two lateral sides along the same direction and then identify the bottom side to the top side along the same direction with a torsion t (see Fig. 2 (left)). Analogously, a *Klein-bottle polyhex* $K(p, q, t)$ is obtained from P by the following boundary identification: first identify two lateral sides along the same direction and then identify the bottom side to the top side along the reverse directions with a torsion t (see Fig. 2 (right)).

For convenience, we adopt the affine coordinate system XOY for $H(p, q, t)$ and $K(p, q, t)$ as introduced in [25,26]: take the bottom side as x-axis and one lateral side as y-axis such that P lies on the non-negative region and the origin O is the intersection of x-axis and y-axis, and define the distance between a pair of parallel edges of a hexagon to be the unit length. According to this affine coordinate system XOY, label each hexagon by its center coordinates (x, y) and denote it by (x, y) or $h_{x,y}$ where $x \in \mathbb{Z}_p$, $y \in \mathbb{Z}_q$ (for any integer n, $\mathbb{Z}_n := \{0, 1, \ldots, n-1\}$). Let e be the upper one of the pair of edges in $h_{x,y}$ perpendicular to y-axis. Color its up-left end by black and another end by white. Then such a 2-coloring gives a bipartition of $H(p, q, t)$ and $K(p, q, t)$. Denote the black end and the white end of e by $b_{x,y}$ and $w_{x,y}$, respectively (see Fig. 3). By this notation, $w_{0,y}$ is adjacent to $b_{0,y}$ and $w_{x,0}$ is adjacent to $b_{x',q-1}$, where $x' = x + t + 1$ for $H(p, q, t)$ and $x' = p - x + t + 1$ for $K(p, q, t)$. For example, $H(6, 3, 2)$ and $K(6, 3, 2)$ arise from the 6×3-parallelogram of a hexagonal lattice in Fig. 3 with torsion $t = 2$. The even cycle $w_{0,y}b_{1,y} \cdots w_{j-1,y}b_{j,y}w_{j,y} \cdots b_{0,y}w_{0,y}$ of $H(p, q, t)$ (resp. $K(p, q, t)$) is also called the *yth layer* of $H(p, q, t)$ (resp. $K(p, q, t)$). Note that, in each yth layer, the vertices incident with an upward vertical edge are black, and the ones incident with a downward vertical edge are white.

Both $H(p, q, t)$ and $K(p, q, t)$ have $2pq$ vertices. None of them with less than 6 vertices is a strong embedding. Clearly, $H(1, 1, 0)$ is the unique toroidal polyhex with 2 vertices; $H(2, 1, 0)$, $H(2, 1, 1)$ and $H(1, 2, 0)$ are the only three toroidal

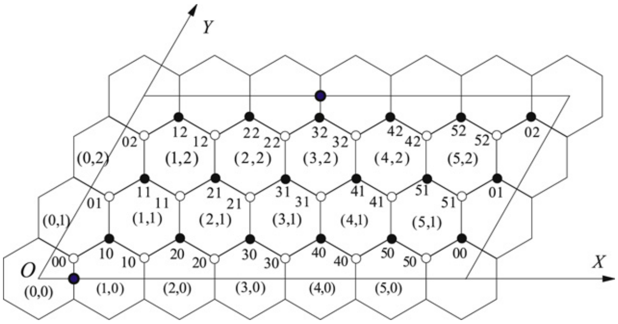

Fig. 3. Affine coordinate system *XOY* of *P* with the vertex labelings.

polyhexes with 4 vertices. Note that $H(1, q, 0), H(p, 1, 0)$ and $H(p, 1, p-1)$ contain a hexagon (i.e. the hexagon $(1, 0)$) which is not bounded by a cycle. So they are not strong embeddings. But a toroidal polyhex $H(p, q, t)$ different from $H(1, q, 0), H(p, 1, 0)$ and $H(p, 1, p-1)$ is clearly a strong embedding. Analogously, $K(1, 1, 0)$ is the unique Klein-bottle polyhex with 2 vertices; $K(2, 1, 0)$, $K(2, 1, 1)$ and $K(1, 2, 0)$ are the only three Klein-bottle polyhexes with 4 vertices. It can be verified that a Klein-bottle polyhex $K(p, q, t)$ is a strong embedding if and only if $\min\{p, q\} \geq 2$. (The hexagon $(0, 0)$ in $K(1, q, 0)$ and the hexagon $(\lfloor \frac{t+1}{2} \rfloor, 0)$ in $K(p, 1, t)$ are not bounded by cycles.)

Let $M_1 := \{w_{j-1,i}b_{j,i} | j \in \mathbb{Z}_p \text{ and } i \in \mathbb{Z}_q\}$, $M_2 := \{w_{j,i}b_{j,i} | j \in \mathbb{Z}_p \text{ and } i \in \mathbb{Z}_q\}$ and $M_3 :=$ {vertical edges of $H(p, q, t)$ (resp. $K(p, q, t)$)}. Then M_1, M_2 and M_3 of $H(p, q, t)$ (resp. $K(p, q, t)$) are pairwise disjoint perfect matchings and compose its edge set.

Theorem 2.1. *Both $H(p, q, t)$ and $K(p, q, t)$ with at least 4 vertices are 1-extendable.* □

An *isomorphism* between two graphs G_1 and G_2 is a bijection $\phi : V(G_1) \to V(G_2)$ such that, for any $u, v \in V(G_1)$, $uv \in E(G_1)$ if and only if $\phi(u)\phi(v) \in E(G_2)$. An *automorphism* of a simple graph G is an isomorphism from G to itself. An automorphism ϕ of G induces a permutation on $E(G)$: $\phi(uv) = \phi(u)\phi(v)$ for $uv \in E(G)$.

For $H(p, q, t)$ and $K(p, q, t)$, define two shifts on their vertex sets as introduced in [25,26]: the *r–l* shift ϕ_{rl} moves every vertex one unit along the reverse direction of the *x*-axis, i.e.,

$$\phi_{rl}(w_{x,y}) = w_{x-1,y} \quad \text{and} \quad \phi_{rl}(b_{x,y}) = b_{x-1,y};$$

and the *t–b* shift ϕ_{tb} moves every vertex one unit along the reverse direction of the *y*-axis, i.e.,

$$\phi_{tb}(w_{x,y}) = w_{x,y-1} \quad \text{and} \quad \phi_{tb}(b_{x,y}) = b_{x,y-1} \quad \text{for } 1 \leq y \leq q-1,$$
$$\phi_{tb}(w_{x,0}) = w_{x',q-1} \quad \text{and} \quad \phi_{tb}(b_{x,0}) = b_{x'',q-1},$$

where $x' = x'' = x + t$ for $H(p, q, t)$ and $x' + 1 = x'' = p - x + t + 1$ for $K(p, q, t)$.

For $H(p, q, t)$, let $\langle \phi_{tb}, \phi_{rl} \rangle$ be the subgroup of the automorphism group generated by ϕ_{tb} and ϕ_{rl}. For each pair of edges $e, e' \in M_i$, e can be transferred to e' by shifts ϕ_{tb} and ϕ_{rl}; that is, there is $\phi \in \langle \phi_{tb}, \phi_{rl} \rangle$ such that $\phi(e) = e'$. We also say that $\langle \phi_{tb}, \phi_{rl} \rangle$ acts transitively on M_i for $i = 1, 2, 3$.

Lemma 2.2. *For each M_i of $H(p, q, t)$, $\langle \phi_{tb}, \phi_{rl} \rangle$ acts transitively on M_i, and $\phi(M_i) = M_i$ for each $\phi \in \langle \phi_{tb}, \phi_{rl} \rangle$.* □

Two graphs embedded on a surface are *equivalent* if there exists a face-preserving isomorphism between them.

Lemma 2.3 ([26]). *ϕ_{rl} and ϕ_{tb} are two hexagon-preserving isomorphisms from $K(p, q, t)$ to $K(p, q, t-2)$ and $K(p, q, t-1)$, respectively.* □

Theorem 2.4 ([26]). *All Klein-bottle polyhexes $K(p, q, t)$ for $t = 0, 1, \ldots, p-1$ are equivalent.* □

By Theorem 2.4, $K(p, q, t)$ can be abbreviated as $K(p, q)$. $K(p, q, t)$ is regarded as a *representation* of $K(p, q)$.

Let $\mathcal{S}$ be a subgraph of $H(p, q, t)$ (resp. $K(p, q)$) such that each component is either a hexagon or an edge with end vertices. Then $\mathcal{S}$ is called an *ideal configuration* [25,26] if $\mathcal{S}$ is alternately incident with white and black vertices along any direction of each layer. An *ideal matching* is an ideal configuration $\mathcal{S}$ without hexagons as components. For example, in Fig. 4, the non-ideal matching on the right side is incident with two consecutive white vertices in the 0th layer. By Lemma 3.1 of Ref. [25] and Lemma 3.2 of Ref. [26], we have the following result.

Lemma 2.5. *An ideal matching $\mathcal{S}$ of $H(p, q, t)$ (resp. $K(p, q)$) can be extended to a perfect matching.* □

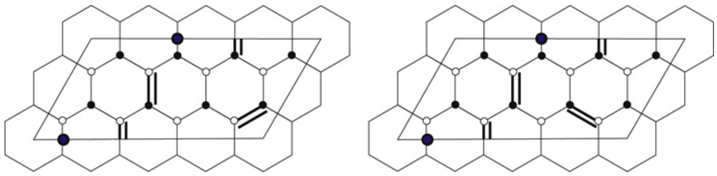

Fig. 4. Matchings (double edges): ideal matching (left) and non-ideal matching (right).

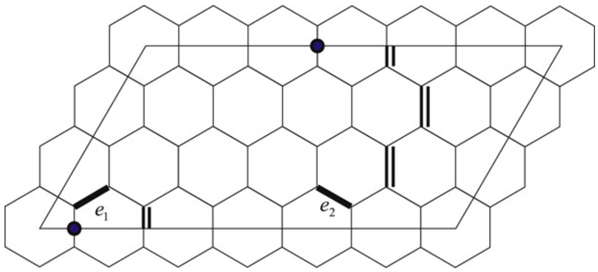

Fig. 5. Illustration for subcase 1.1 in the proof of Lemma 3.1.

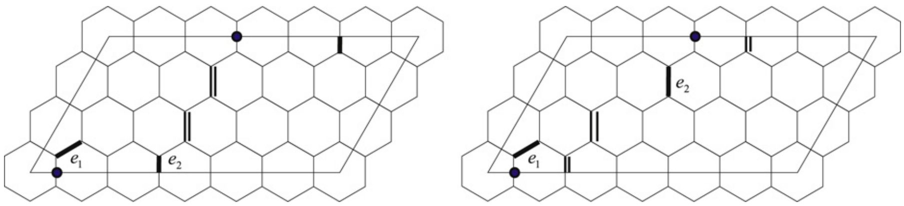

Fig. 6. Illustration for subcase 1.2 in the proof of Lemma 3.1.

3. Toroidal polyhexes $H(p, q, t)$

In this section, we consider the 2-extendability of $H(p, q, t)$.

Lemma 3.1. *For* $\min(p, q) \geq 2$, *$H(p, q, t)$ is a brace.*

Proof. Let $H(p, q, t)$ be a toroidal polyhex with $\min(p, q) \geq 2$. Then $H(p, q, t)$ has at least 8 vertices. Let e_1, e_2 be any two independent edges of $H(p, q, t)$. Suppose $e_1 \in M_i$, $e_2 \in M_j$ $(i, j \in \{1, 2, 3\})$. If $i = j$, then M_i is a perfect matching containing both e_1 and e_2. So we may assume that $i \neq j$. In the following, we will construct an ideal matching $\mathcal{S}$ containing both e_1 and e_2. Then by Lemma 2.5, $H(p, q, t)$ has a perfect matching M containing $e_1, e_2 \in \mathcal{S}$. That is $H(p, q, t)$ is a brace.

Case 1. $e_1 \in M_1$ and $e_2 \notin M_1$. According to Lemma 2.2, we may assume $e_1 = w_{0,0}b_{1,0}$.

Subcase 1.1. $e_2 = w_{x_2,y_2}b_{x_2,y_2} \in M_2$.

For $y_2 \neq 0$, let $E_0 := \{w_{x,0}b_{x+1,0} | x \in \mathbb{Z}_p\}$ and $E_y := \{w_{x,y}b_{x,y} | x \in \mathbb{Z}_p\}$ for $y \neq 0$. Then $M = \cup_{y \in \mathbb{Z}_q} E_y$ is a perfect matching containing e_1 and e_2.

If $y_2 = 0$, then $2 \leq x_2 \leq p - 1$ since e_1 and e_2 are disjoint. Choose a series of vertical edges (see Fig. 5):

$$w_{1,0}b_{2+t,q-1} \quad \text{and} \quad w_{x_2,y}b_{x_2+1,y-1} \quad \text{for } y = 1, 2, \ldots, q-1.$$

Let $\mathcal{S} := \{w_{x_2,y}b_{x_2+1,y-1} | y \in \mathbb{Z}_q \setminus \{0\}\} \cup \{w_{1,0}b_{2+t,q-1}, e_1, e_2\}$. Then $\mathcal{S}$ is an ideal matching since it is incident with the vertices in the 0th layer as ordered $w_{0,0}, b_{1,0}, w_{1,0}, b_{x_2,0}, w_{x_2,0}$ and $b_{x_2+1,0}$, and $w_{x_2,y}, b_{x_2+1,y}$ in the yth layer $(1 \leq y \leq q - 2)$, and $w_{x_2,q-1}, b_{2+t,q-1}$ in the $(q - 1)$th layer.

Subcase 1.2. $e_2 \in M_3$. Assume that $e_2 = w_{x_2,y_2}b_{x_2+1,y_2-1}$ if $y_2 \neq 0$, and $e_2 = w_{x_2,0}b_{x_2+t+1,q-1}$, otherwise.

Choose a series of vertical edges (see Fig. 6):

$$w_{1,y}b_{2,y-1} \quad \text{for } y = 1, 2, \ldots, y_2 - 1, \quad \text{and} \quad w_{x_2,y}b_{x_2+1,y-1} \quad \text{for } y = y_2 + 1, \ldots, q - 1.$$

Let $E := \{w_{1,y}b_{2,y-1} | 1 \leq y \leq y_2 - 1\} \cup \{w_{x_2,y}b_{x_2+1,y-1} | y_2 + 1 \leq y \leq q - 1\}$.

If $y_2 = 0$, then $x_2 \neq 0$ since e_1 and e_2 are disjoint. Note that $E = \{w_{x_2,y}b_{x_2+1,y-1} | y_2 + 1 \leq y \leq q - 1\}$. Then $\mathcal{S} = E \cup \{e_1, e_2\}$ is an ideal matching since it is incident with the vertices in the 0th layer as ordered $w_{0,0}, b_{1,0}, w_{x_2,0}$ and $b_{x_2+1,0}$, and two vertices with different colors in all other yth layers (see Fig. 6 (left)).

Suppose $y_2 \neq 0$. Note that $x_2 \neq 0$ if $y_2 = 1$. Let $\mathcal{S} := E \cup \{w_{1,0}b_{t+2,q-1}, e_1, e_2\}$. Then $\mathcal{S}$ is an ideal matching since it is incident with the vertices in the 0th layer as ordered $w_{0,0}, b_{1,0}, w_{1,0}$ and $b_{x,0}$ ($x = 2$ if $y_2 \neq 1$, and $x = x_2$, otherwise), and two vertices with different colors in all other yth layers (see Fig. 6 (right)).

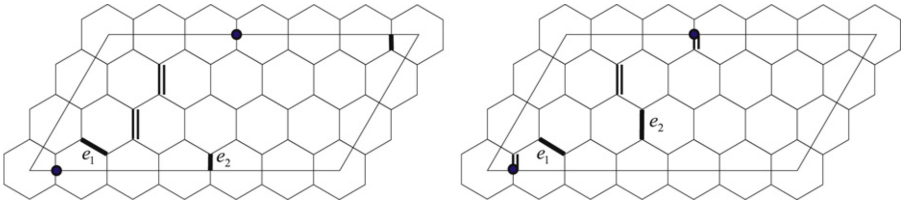

Fig. 7. Illustration for Case 2 in the proof of Lemma 3.1.

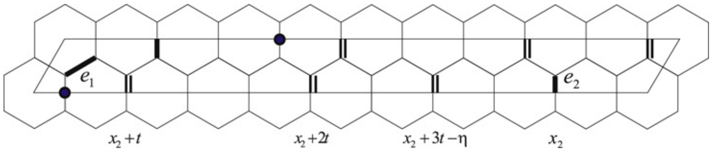

Fig. 8. Illustration for Subcase 1.2 in the proof of Lemma 3.2.

Case 2. $e_1 \in M_2$ and $e_2 \in M_3$. By Lemma 2.2, we may assume that $e_1 = w_{1,0}b_{1,0}$. Let $e_2 = w_{x_2,y_2}b_{x_2+1,y_2-1}$ if $y_2 \neq 0$, and $e_2 = w_{x_2,0}b_{x_2+t+1,q-1}$, otherwise.

Choose a series of vertical edges (see Fig. 7):

$$w_{1,y}b_{2,y-1} \quad \text{for } y = 1, 2, \ldots, y_2 - 1, y_2 + 1, \ldots, q - 1.$$

If $y_2 = 0$, then $x_2 \neq 1$. Let $\mathcal{S} =: \{w_{1,y}b_{2,y-1} | y \in \mathbb{Z}_q \setminus \{0\}\} \cup \{e_1, e_2\}$. Then $\mathcal{S}$ is an ideal matching since it is incident with $b_{1,0}, w_{1,0}, b_{2,0}$ and $w_{x_2,0}$ in the 0th layer, and two vertices with different colors in all other yth layers (see Fig. 7 (left)).

Suppose $y_2 \neq 0$. Note that $x_2 \neq 0$ if $y_2 = 1$. Let $\mathcal{S} := \{w_{1,y}b_{2,y-1} | y \in \mathbb{Z}_q \setminus \{0, y_2\}\} \cup \{w_{0,0}b_{t+1,q-1}, e_1, e_2\}$. Then $\mathcal{S}$ is an ideal matching since it is incident with the vertices in the 0th layer as ordered $b_{1,0}, w_{1,0}, b_{x,0}$ and $w_{0,0}$ ($x = 2$ if $y_2 \neq 1$, and $x = x_2$, otherwise), and two vertices with different colors in all other yth layers (see Fig. 7 (right)). □

Lemma 3.2. *Suppose* $\min(p, q) = 1$ *and* $H(p, q, t)$ *has at least* 6 *vertices. Then* $H(p, q, t)$ *is a brace if and only if* $p \geq 3$ *and* $t \neq 0, p - 1$.

Proof. For convenience, we omit the second label 0's of all vertices of $H(p, 1, t)$ ($p \geq 3$), i.e. $w_x = w_{x,0}$, $b_x = w_{x,0}$. Then w_x is adjacent to b_{x+t+1}. For $a, b \in \mathbb{Z}_p$ and $a < b$, let $[a, b] := \{a, a + 1, a + 2, \ldots, b - 1, b\}$ be the interval of $\mathbb{Z}_p$ between a and b with the increasing order. Then $[b, a] := \{b, b + 1, b + 2, \ldots, a - 1, a\} = \mathbb{Z}_p \setminus [a + 1, b - 1]$.

Necessary: It suffices to prove that $H(1, q, 0)$ ($q \geq 3$) and $H(p, 1, t)$ ($p \geq 3$) with $t = 0, p - 1$ are not 2-extendable. For $H(1, q, 0)$ ($q \geq 3$), choose two independent edges $e_1 = w_{0,0}b_{0,0}$ and $e_2 = w_{0,2}b_{0,1}$. Then $w_{0,1}$ is an isolated vertex in $H(1, q, 0) - \{w_{0,0}, b_{0,0}, b_{0,1}, w_{0,2}\}$. For $H(p, 1, t)$ ($p \geq 3$) with $t = 0, p - 1$, choose two independent edges $e_1 = w_0b_1$ and $e_2 = b_2w_2$. Then w_1 is an isolated vertex in $H(p, 1, t) - \{w_0, b_0, b_1, w_2\}$. Hence $H(1, q, 0)$ and $H(p, 1, t)$ with $t = 0, p - 1$ are not 2-extendable.

Sufficiency: It suffices to prove that $H(p, 1, t)$ with $1 \leq t \leq p-2$ is 2-extendable. Let e_1, e_2 be any two independent edges. Assume $e_1 \in M_i, e_2 \in M_j$ ($i, j \in \{1, 2, 3\}$). If $i = j$, then $e_1, e_2 \in M_i$. If $i \neq j$, we will construct an ideal matching $\mathcal{S}$ such that $e_1, e_2 \in \mathcal{S}$. By Lemma 2.5, $H(p, 1, t)$ has a perfect matching M such that $\mathcal{S} \subseteq M$. Further $e_1, e_2 \in M$.

Case 1. $e_1 \in M_1$ and $e_2 \in M_2$ or $e_2 \in M_3$. By Lemma 2.2, we may assume $e_1 = w_0b_1$.

Subcase 1.1. $e_2 = w_{x_2}b_{x_2} \in M_2$. Then $2 \leq x_2 \leq p - 1$ since e_1 and e_2 are disjoint.

Consider a vertical edge w_xb_{x+t+1} such that $1 \leq x \leq x_2 - 1$. Then $t+2 \leq x+t+1 \leq x_2+t$. Since $2+t \leq p$ and $x_2+1 \leq x_2+t$, $[t + 2, x_2 + t] \cap [x_2 + 1, p] \neq \emptyset$. Let $x_0 + t + 1 \in [t + 2, x_2 + t] \cap [x_2 + 1, p]$. Then $x_0 \in [1, x_2 - 1]$ and $x_0 + t + 1 \in [x_2 + 1, p]$. Choose the additional edge $w_{x_0}b_{x_0+t+1}$. Then $\mathcal{S} = \{e_1, e_2, w_{x_0}b_{x_0+t+1}\}$ is an ideal matching since it is incident with the vertices in the 0th layer as ordered $w_0, b_1, w_{x_0}, b_{x_2}, w_{x_2}$ and b_{x_0+t+1}.

Subcase 1.2. $e_2 = w_{x_2}b_{x_2+t+1} \in M_3$. Then $1 \leq x_2 \leq p - 1$ and $x_2 + t + 1 \not\equiv 1 \pmod{p}$.

If $x_2 + t + 1 \leq p$, then $\mathcal{S} = \{e_1, e_2\}$ is an ideal matching since $\mathcal{S}$ is incident with the vertices in the 0th layer as ordered w_0, b_0, w_{x_2} and b_{x_2+t+1}.

If $x_2 + t + 1 > p$, then it is obvious that $\{e_1, e_2\}$ is not an ideal matching. Choose a series of vertical edges:

$$w_{x_2+t}b_{x_2+2t+1}, \ldots, w_{x_2+(j-1)t}b_{x_2+jt+1},$$

such that j is maximal subject to $x_2 + jt + 1 - p \leq x_2$ (see Fig. 8, where $j = 3$). Then $x_2 + (j + 1)t - p \in [x_2, x_2 + t + 1]$.

If $x_2 + (j+1)t + 1 - p \leq p$, let $\mathcal{S} := \{w_{x_2+(r-1)t}b_{x_2+rt+1} | r = 1, \ldots, j + 1\} \cup \{e_1, e_2\}$. Then $\mathcal{S}$ is incident with the vertices in the 0th layer as ordered $w_0, b_1, \ldots, w_{x_2+rt}, b_{x_2+rt+1}, \ldots, w_{x_2}$ and $b_{x_2+(j+1)t+1}$ since $x_2 + jt + 1 - p \leq x_2 < x_2 + (j + 1)t + 1 - p \leq p$. Hence $\mathcal{S}$ is an ideal matching.

So suppose $x_2 + (j+1)t + 1 - p > p$. Let $\eta := x_2 + (j+1)t + 1 - 2p$ (see Fig. 8, where $\eta = 1$). Then $\eta < x_2 + t + 1 - p \leq t - 1$. Choose the edge $w_{x_2+jt-\eta}b_{x_2+(j+1)t+1-\eta}$. Then $x_2 + (j + 1)t + 1 - \eta = 2p$ and $x_2 + (j - 1)t + 1 \leq x_2 + jt - \eta$. Let

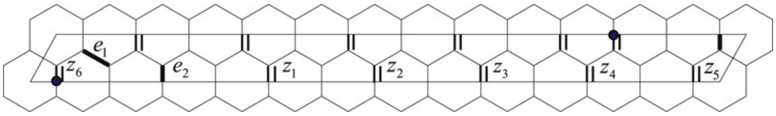

Fig. 9. Toroidal polyhex $H(13, 1, 10)$ with $t = 5(p-1)/6$.

$\mathcal{S} := \{w_{x_2+(r-1)t}b_{x_2+rt+1} | r = 1, \ldots, j\} \cup \{w_{x_2+jt-\eta}b_0, e_1, e_2\}$. Hence $\mathcal{S}$ is incident with vertices in the 0th layer as ordered $w_0, b_1, w_{x_2+t}, b_{x_2+t+1}, \ldots, w_{x_2+(j-1)t}, b_{x_2+(j-1)t+1}, w_{x_2+jt-\eta}, b_{x_2+jt+1}, w_{x_2}$ and b_0. Further $\mathcal{S}$ is an ideal matching.

Subcase 2. $e_1 \in M_2$ and $e_2 \in M_3$. By Lemma 2.2 we may assume that $e_1 = w_1b_1$ and $e_2 = w_{x_2}b_{x_2+t+1}$. Note that $x_2 \neq 1$ and $x_2 + t + 1 \neq p + 1$ since e_1 and e_2 are disjoint (see Fig. 9).

If $x_2 + t + 1 > p + 1$, then $\mathcal{S} = \{e_1, e_2\}$ is an ideal matching since it is incident with b_1, w_1, b_{x_2+t+1} and w_{x_2} in the 0th layer.

So suppose $x_2 + t + 1 \leq p$. Consider a vertical edge w_xb_{x+t+1} with $2 \leq x + t + 1 - p \leq x_2$ (i.e. $p - t + 1 \leq x \leq x_2 + p - t - 1$).

For $x_2 + p - t - 1 \geq x_2 + t + 1$ (i.e. $t \leq \frac{p-2}{2}$), let $z_1 := x_2 + p - t - 1$. Choose the edge $w_{z_1}b_{z_1+t+1} = w_{z_1}b_{x_2}$. Then $\mathcal{S} = \{e_1, e_2, w_{z_1}b_{x_2}\}$ is an ideal matching since it is incident with vertices in the 0th layer as ordered $b_1, w_1, b_{x_2}, w_{x_2}, b_{x_2+t+1}$ and w_{z_1}.

So suppose $t \geq \frac{p-1}{2}$. Clearly, $\{e_1, e_2, w_{z_1}b_{x_2}\}$ is not an ideal matching. So we will construct a series of additional vertical edges to obtain an ideal matching.

Let $\varepsilon := p - 1 - t$ and N be the minimal positive integer to guarantee that $(p-1) - \frac{N-1}{N}(p-1) < \varepsilon$. Then $t < \frac{N-1}{N}(p-1)$ and $t \geq \frac{N-2}{N-1}(p-1)$. Choose a series of vertical edges:

$$w_{z_1}b_{z_1+t+1}, \ldots, w_{z_r}b_{z_r+t+1}, \ldots, w_{z_{N-2}}b_{z_{N-2}+t+1},$$

such that $z_r = x_2 + r\varepsilon$ for $r = 1, \ldots, N-2$ (see Fig. 9, $N = 7$ for $H(13, 1, 10)$). Then $z_1 + t + 1 \equiv x_2 \pmod{p}$ and $z_r + t + 1 = x_2 + r\varepsilon + t + 1 = z_{r-1} + p \equiv z_{r-1} \pmod{p}$ for $r = 2, \ldots, N-2$. On the other hand,

$$\begin{aligned} z_{N-2} &= x_2 + (N-2)(p-t-1) \\ &= x_2 + t + (N-2)(p-1) - (N-1)t \\ &\leq x_2 + t + (N-2)(p-1) - (N-1)\frac{(N-2)(p-1)}{N-1} \\ &= x_2 + t. \end{aligned}$$

Hence $z_1 < \cdots < z_{r-1} < z_r < \cdots < z_{N-2} < x_2 + t + 1 \leq p$.

Now, consider a vertical edge w_xb_{x+t+1} with $x + t + 1 \in [z_{N-3} + 1, z_{N-2}]$ (i.e., $z_{N-2} + 1 \leq x \leq x_2 + (N-1)(p-t-1)$). Since $x_2 + t + 1 \leq p$, $z_{N-2} + 1 \leq p$. Clearly, we also have inequality $x_2 + (N-1)(p-t-1) \geq x_2 + t + 1$ since $t < \frac{N-1}{N}(p-1)$. So we can choose $z_{N-1} \in [z_{N-2} + 1, x_2 + (N-1)(p-t-1)] \cap [x_2 + t + 1, p] \neq \emptyset$. Then $z_{N-3} + 1 \leq z_{N-1} + t + 1 - p \leq z_{N-2}$.

Let $\mathcal{S} := \{w_{z_r}b_{z_r+t+1} | r = 1, \ldots, N-2\} \cup \{e_1, e_2, w_{z_{N-1}}b_{z_{N-1}+t+1}\}$. Then $\mathcal{S}$ is incident with the vertices in the 0th layer as ordered $b_1, w_1, b_{z_1+t+1}, w_{x_2}, b_{z_2+t+1}, w_{z_1}, \ldots, b_{z_{N-2}+t+1}, w_{z_{N-3}}, b_{z_{N-1}+t+1}, w_{z_{N-2}}, b_{x_2+t+1}$ and $w_{z_{N-1}}$ since $0 < 1 < z_1 + t + 1 = x_2 < z_2 + t + 1 = z_1 < \cdots < z_{N-2} + t + 1 = z_{N-3} < z_{N-1} + t + 1 \leq z_{N-2} < x_2 + t + 1 \leq z_{N-1} \leq p$. So $\mathcal{S}$ is an ideal matching. □

By Lemmas 3.1 and 3.2, we have the following result.

Theorem 3.3. *Let $H(p, q, t)$ be a toroidal polyhex with at least 6 vertices. Then $H(p, q, t)$ is a brace if and only if one of the following cases appears:*

(1) $\min\{p, q\} \geq 2$;
(2) $q = 1, p \geq 3$ *and* $t \neq 0, p-1$. □

Theorem 3.3 implies that Theorem 1.1 holds for toroidal polyhexes.

4. Klein-bottle polyhexes $K(p, q)$

We now turn to discuss the 2-extendability of $K(p, q)$. Note that every white vertex $w_{x,0}$ in the 0th layer is adjacent to a black vertex $b_{p-x+t+1,q-1}$ in the $(q-1)$th layer in $K(p, q, t)$, a representation of $K(p, q)$.

Lemma 4.1. *For* $\min(p, q) = 1$, *$K(p, q)$ is not a brace.*

Proof. Let $K(p, q)$ be a Klein-bottle polyhex with at least 6 vertices and $\min(p, q) = 1$. Then $p \geq 3$ if $q = 1$ and $q \geq 3$ if $p = 1$.

Suppose that $q = 1$ and $p \geq 3$. It suffices to consider $K(p, 1, 1)$ by Theorem 2.4. The neighbors of $w_{1,0}$ are $b_{p-1+t+1,0}, b_{1,0}$ and $b_{2,0}$. Since $b_{p-1+t+1,0} = b_{1,0}$, $w_{1,0}$ is an isolated vertex in $K(p, q) - \{w_{0,0}, b_{1,0}, w_{2,0}, b_{2,0}\}$ (see Fig. 10). Hence two disjoint edges $w_{0,0}b_{1,0}$ and $w_{2,0}b_{2,0}$ can not be extended to a perfect matching of $K(p, 1, 1)$. So $K(p, 1)$ is not 2-extendable.

Now suppose that $p = 1$ and $q \geq 3$. Consider two disjoint edges $w_{0,0}b_{0,0}$ and $w_{0,2}b_{0,1}$. Since $K(p, q) - \{w_{0,0}, b_{0,0}, w_{0,2}, b_{0,1}\}$ has an isolated vertex $w_{0,1}$, $K(1, q)$ is not 2-extendable. □

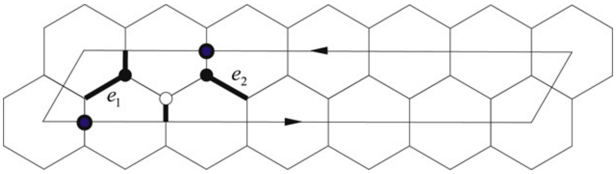

Fig. 10. Klein-bottle polyhex $K(6, 1, 1)$.

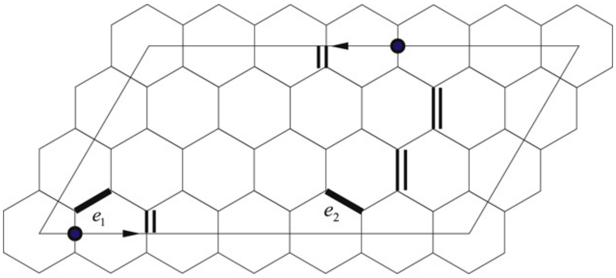

Fig. 11. Illustration for Case 1 in the proof of Lemma 4.2.

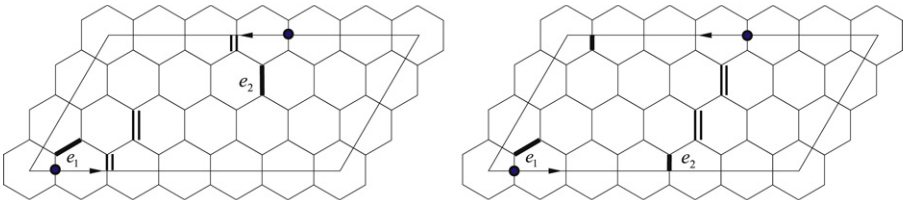

Fig. 12. Illustration for Case 2 in the proof of Lemma 4.2.

Lemma 4.2. *For* $\min(p, q) \geq 2$, $K(p, q)$ *is a brace.*

Proof. Let $K(p, q)$ be a Klein-bottle polyhex with $\min(p, q) \geq 2$. It suffices to prove that any two independent edges e and e' of $K(p, q, 0)$ can be extended to a perfect matching. If $\{e, e'\} \subset M_i$, it is true trivially. Otherwise, by applying repeatedly the shifts ϕ_{rl} and ϕ_{tb} if necessary, e and e' of $K(p, q, 0)$ can be transformed to independent edges e_1 and e_2 of some $K(p, q, t)$ for $t \in \mathbb{Z}_p$ such that e_1 is incident with $b_{1,0}$. So it needs to show that e_1 and e_2 can be extended to a perfect matching for every representation $K(p, q, t)$. By Lemma 2.5, we will construct an ideal matching $\mathcal{S}$ containing e_1 and e_2. There are three cases to be considered.

Case 1. $e_1 = w_{0,0}b_{1,0} \in M_1$ and $e_2 = w_{x_2,y_2}b_{x_2,y_2} \in M_2$.

If $y_2 \neq 0$, let $E_0 := \{w_{x,0}b_{x+1,0} | x \in \mathbb{Z}_p\}$ and $E_y := \{w_{x,y}b_{x,y} | x \in \mathbb{Z}_p\}$ for $y \neq 0$. Then $e_1 \in E_0$ and $e_2 \in E_{y_2}$. So $M = \cup_{y \in \mathbb{Z}_q} E_y$ is a perfect matching containing e_1 and e_2.

Now suppose $y_2 = 0$. Note that $2 \leq x_2 \leq p - 1$ since e_1 and e_2 are disjoint. Choose a series of vertical edges:

$$w_{1,0}b_{t,q-1} \quad \text{and} \quad w_{x_2,y}b_{x_2+1,y-1} \quad \text{for } y = 1, 2, \ldots, q - 1.$$

Let $\mathcal{S} := \{w_{x_2,y}b_{x_2+1,y-1} | 1 \leq y \leq q - 1\} \cup \{w_{1,0}b_{t,q-1}, e_1, e_2\}$ (see Fig. 11). Then $\mathcal{S}$ is an ideal matching since it is incident with the vertices in the 0th layer as ordered $w_{0,0}, b_{1,0}, w_{1,0}, b_{x_2,0}, w_{x_2,0}$ and $b_{x_2+1,0}$, and $w_{x_2,y}, b_{x_2+1,y}$ in the yth layer ($y \neq 0, p - 1$), and $w_{x_2,q-1}, b_{t,q-1}$ in the $(q - 1)$th layer.

Case 2. $e_1 = w_{0,0}b_{1,0} \in M_1$ and $e_2 \in M_3$. We may assume that $e_2 = w_{x_2,y_2}b_{x_2+1,y_2-1}$ if $y_2 \neq 0$, $e_2 = w_{x_2,0}b_{p-x_2+t+1,q-1}$, otherwise.

First suppose $y_2 \neq 0$. Note that $x_2 + 1 \neq 1$ if $y_2 = 1$ since e_1 and e_2 are disjoint. Choose a series of vertical edges:

$$w_{1,0}b_{t,q-1} \quad \text{and} \quad w_{1,y}b_{2,y-1} \quad \text{for } y = 1, 2, \ldots, y_2 - 1, y_2 + 1, \ldots, q - 1.$$

Let $\mathcal{S} := \{w_{1,y}b_{2,y-1} | y \in \mathbb{Z}_q \setminus \{0, y_2\}\} \cup \{w_{1,0}b_{t,q-1}, e_1, e_2\}$ (see Fig. 12 (left)). Then $\mathcal{S}$ is an ideal matching since it is incident with the vertices in the 0th layer as ordered $w_{0,0}, b_{1,0}, w_{1,0}$ and $b_{x,0}$ ($x = 1$ if $y_2 \neq 1$, and $x = x_2$, otherwise), and two vertices with two different colors in all other yth layers.

If $y_2 = 0$, then $x_2 \neq 0$ since e_1 and e_2 are disjoint. Choose a series of vertical edges:

$$w_{x_2,y}b_{x_2+1,y-1} \quad \text{for } y = 1, 2, \ldots, q - 1.$$

Let $\mathcal{S} := \{w_{x_2,y}b_{x_2+1,y-1} | y \in \mathbb{Z}_q \setminus \{0\}\} \cup \{e_1, e_2\}$ (see Fig. 12 (right)). Then $\mathcal{S}$ is an ideal matching since it is incident with $w_{0,0}, b_{1,0}, w_{x_2,0}$ and $b_{x_2+1,0}$ in the 0th layer, and two vertices with different colors in all other yth layers.

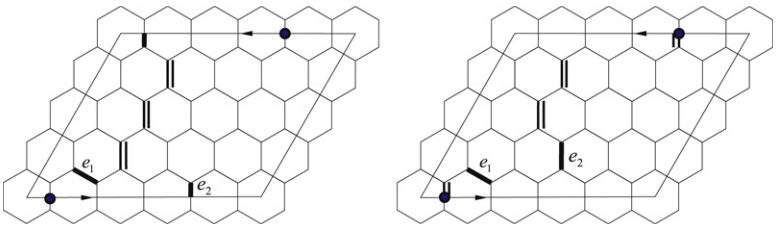

Fig. 13. Illustration for Case 3 in the proof of Lemma 4.2.

Case 3. $e_1 = w_{1,0}b_{1,0} \in M_2$ and $e_2 \in M_3$. Assume that $e_2 = w_{x_2,y_2}b_{x_2+1,y_2-1}$ if $y_2 \neq 0$, and $e_2 = w_{x_2,0}b_{p-x_2+t+1,q-1}$, otherwise.

First choose a series of vertical edges:

$$w_{1,y}b_{2,y-1} \quad \text{for } y = 1, 2, \dots, y_2 - 1, y_2 + 1, \dots, q - 1.$$

If $y_2 = 0$, then $x_2 \neq 1$. Let $\mathcal{S} := \{w_{1,y}b_{2,y-1} | y \in \mathbb{Z}_p \setminus \{0\}\} \cup \{e_1, e_2\}$ (see Fig. 13 (left)). Then $\mathcal{S}$ is an ideal matching since it is incident with $b_{1,0}, w_{1,0}, b_{2,0}, w_{x_2,0}$ in the 0th layer, and two vertices with different colors in all other yth layers.

So suppose $y_2 \neq 0$. Note that $x_2 \neq 0$ if $y_2 = 1$. Let $\mathcal{S} := \{w_{1,y}b_{2,y-1} | y \in \mathbb{Z}_p \setminus \{0, y_2\}\} \cup \{w_{0,0}b_{t+1,q-1}, e_1, e_2\}$ (see Fig. 13 (right)). Then $\mathcal{S}$ is incident with the vertices in the 0th layer as ordered $b_{1,0}, w_{1,0}, b_{x,0}$ and $w_{0,0}$ ($x = x_2 + 1$ if $y_2 = 1$, and $x = 2$, otherwise), and two vertices with different colors in all other yth layers. So $\mathcal{S}$ is a desired ideal matching. $\square$

By Lemmas 4.1 and 4.2, we immediately have the following theorem.

Theorem 4.3. *A Klein-bottle polyhex $K(p, q)$ is a brace if and only if* $\min(p, q) \geq 2$. $\square$

Theorem 4.3 implies that Theorem 1.1 is true for Klein-bottle polyhexes. Together with the result in Section 3, Theorem 1.1 follows immediately.

Acknowledgements

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